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LAMINAR BUOYANCY-INDUCED AXISYMMETRIC FREE BOUNDARY FLOWS IN A THERMALLY STRATIFIED MEDIUM

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Abstract—An analytical study of the laminar flow in axisymmetric thermal plumes and buoyant jets in stably stratified media is carried out. A linear variation of the ambient temperature is considered in detail, with the input of thermal energy and momentum specified over a small finite region. The downstream flow is studied in detail and numerical results are obtained over a wide range of stratification level and for various values of the governing parameters. Of particular interest were the downstream variation of the centerline velocity and temperature. The corresponding profiles are also obtained. A stable stratification is found to cause a decrease in the buoyancy and, hence, in the velocity. A temperature defect arises in the outer region of the boundary layer and the velocity level in the flow also decreases substantially. The effect is found to increase with the downstream distance and with an increase in the stratification level. Buoyancy decreases downstream and the observed results indicate the nature of flow in the region where the buoyancy force becomes zero and then negative. The underlying physical processes are considered in terms of the results obtained.

NOMENCLATURE

d, radius of finite region over which input conditions are specified;
g, acceleration due to gravity;
Gr, Grashof number for a buoyant jet, defined in equation (5);
Pr, Prandtl number;
Re, Reynolds number for a buoyant jet, defined in equation (5);
S, stratification parameter, defined in equation (9);
S_0, $S = S_0X^{n-1}$, if ambient temperature varies as X^n;
t, local temperature in the flow;
t_c, centerline temperature;
t_0, temperature at the input location, $x = 0$;
$t_{\infty,x}$, local ambient temperature;
$t_{\infty,0}$, ambient temperature at $x = 0$;
Δt, temperature excess at $x = 0$, $\Delta t = t_0 - t_{\infty,0}$;
u, vertical velocity component;
U, dimensionless vertical velocity;
U_c, dimensionless centerline vertical velocity;
U_0, vertical velocity at $x = 0$;
v, transverse velocity component;
V, dimensionless transverse velocity;
x, vertical coordinate distance;
X, dimensionless vertical coordinate, defined in equations (2) and (5a);
y, radial coordinate distance;
Y, dimensionless radial coordinate, defined in equations (2) and (5a);
$\tilde{Y}$, dimensionless radial coordinate, defined in equation (5b).

Greek symbols

α, thermal diffusivity of the fluid;
β, coefficient of volumetric thermal expansion of the fluid;
ν, kinematic viscosity;
θ, dimensionless temperature;
θ_c, dimensionless centerline temperature;
τ, physical time;
τ^*, dimensionless time.

INTRODUCTION

IN MANY natural and mixed convection flows of interest, in technology and in nature, the ambient fluid is stably stratified, with lighter fluid overlying denser fluid. Frequently, this density distribution is due to a temperature variation with height and a specified buoyancy-induced flow in the thermally stratified environment is to be determined. Thermal stratification is commonly encountered in the atmosphere and in water bodies, such as lakes and cooling ponds. Heat rejection from power plants and other industrial systems, therefore, often involves natural and mixed convection flows in stratified media [1, 2]. Though most of these flows are turbulent, heat removal processes in technology and energy storage in fluids often involve laminar flow in thermally stratified media. Heat transfer processes in enclosures often give rise to thermal stratification, which is also of concern in many experimental studies of heat transfer [3].

In most applications of practical interest, the thermal stratification is a stable one, which requires that the ambient temperature decrease with height at a rate less than the adiabatic lapse rate, for a fluid whose density decreases with an increase in temperature [4]. Though the adiabatic lapse rate is of particular interest for heat transfer processes in nature, a stable thermal stratification resulting from a temperature increase with height is of much greater interest and importance in technology. In this study, an ambient temperature increase with height is considered, the stratification level being determined by the rate of temperature rise.

Natural convection flows in thermally stratified media have been of interest to several investigators. The flow due to a heated surface has been investigated experimentally and analytically in several studies [3, 5, 6]. Turbulent free boundary flows, such as those due to thermal plumes and buoyant jets, are of particular interest in heat rejection and environmental problems. Analytical studies, employing integral and other approximate methods, as well as experimental investigations, have been carried out to determine the downstream variation of the flow and the height to which the flow rises [7, 8].

The present work considers laminar axisymmetric free boundary flows, particularly the axisymmetric buoyant jet and the axisymmetric thermal plume, in thermally stratified media. The problem is approached analytically as a vertical buoyancy-induced boundary layer problem, with a specified ambient temperature variation. Such flows are of interest in heat transfer from heated bodies in enclosures, in heat rejection and energy input into enclosed regions, in several experimental studies of heat transfer and in energy storage as sensible heat in a fluid. Laminar axisymmetric buoyant jets and point source plumes in an isothermal environment have been studied analytically by several workers [9–13]. Most of the relevant experimental studies for stratified media have been carried out for turbulent flows and for concentration stratified environments. The effect of a stable density variation in the ambient medium on these flows has not been considered analytically in detail, though the importance of thermal stratification in other natural convection flows has been demonstrated. The problem is an important one, being related to several industrial heat transfer processes.

Most of the analytical studies of natural convection in thermally stratified media have been carried out by means of the similarity analysis, which converts the governing partial differential equations into ordinary differential equations, or by employing various approximate analyses for turbulent flows. The present work solves the governing boundary layer equations by finite difference techniques, since the similarity analysis is very restrictive in its applicability. Though various ambient temperature distributions were considered, attention was focused on the linear profile, because an arbitrarily stratified environment tends to become linearly stratified as time elapses, due to the underlying conductive mechanisms [14], and because the linear profile is the most frequently encountered one in practice. Prandtl number values of 7.0 and 0.7, corresponding to water and air at room temperature, were considered. The velocity and the temperature fields that arise are studied as functions of the input parameters, such as the stratification level and the given velocity and temperature at the source. Of particular interest was the downstream variation of the temperature and the velocity fields. Several very interesting features are observed and these are considered in terms of the basic physical mechanisms underlying these flows. The centerline velocity and temperature variation with height were determined, indicating a pronounced effect of thermal stratification on the flow. The results are also compared with earlier studies for various other natural convection flows and interesting differences are observed.

ANALYSIS AND NUMERICAL SCHEME

The boundary layer equations for a laminar vertical axisymmetric natural convection flow may be written, employing the Boussinesq approximations for the density variation and assuming the other properties to be constant. The viscous dissipation and pressure work terms in the energy equation are taken as negligible. The governing equations are:

$$\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(yv) = 0, \tag{1a}$$

$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\nu}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right) + g\beta(t - t_{\infty,x}), \tag{1b}$$

$$\frac{\partial t}{\partial \tau} + u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{\alpha}{y}\frac{\partial}{\partial y}\left(y\frac{\partial t}{\partial y}\right). \tag{1c}$$

For an axisymmetric plume, the only input is a specified temperature or heat input at $x = 0$. Therefore, a convection velocity is employed to nondimensionalize the velocity components and the other nondimensional variables are given as:

$$X = x\left(\frac{g\beta\Delta t}{\nu^2}\right)^{1/3}, \quad Y = y\left(\frac{g\beta\Delta t}{\nu^2}\right)^{1/3},$$

$$\theta = \frac{t - t_{\infty,x}}{t_0 - t_{\infty,0}}, \quad U = u/(\nu g\beta\Delta t)^{1/3},$$

$$V = v/(\nu g\beta\Delta t)^{1/3}, \quad \tau^* = \frac{\tau}{\nu^{1/3}}(g\beta\Delta t)^{2/3}. \tag{2}$$

The dimensionless governing equations are then obtained as:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{V}{Y} = 0, \tag{3a}$$

$$\frac{\partial U}{\partial \tau^*} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \theta + \frac{\partial^2 U}{\partial Y^2} + \frac{1}{Y}\frac{\partial U}{\partial Y}, \tag{3b}$$

$$\frac{\partial\theta}{\partial\tau^*} + U\frac{\partial\theta}{\partial X} + V\frac{\partial\theta}{\partial Y} + \frac{U}{\Delta t}\frac{\mathrm{d}t_{\infty,x}}{\mathrm{d}X} = \frac{1}{Pr}\left(\frac{\partial^2\theta}{\partial Y^2} + \frac{1}{Y}\frac{\partial\theta}{\partial Y}\right) \quad (3c)$$

with the boundary and initial conditions:

$\tau^* > 0$:

$X = 0$; $U = 0$ for $Y \geq 0$, $\theta = 1.0$ for $Y \leq d$ and $\theta = 0.0$ for $Y > d$;

$$X > 0;\ \frac{\partial U}{\partial Y} = \frac{\partial\theta}{\partial Y} = V = 0 \text{ at } Y = 0; \quad (4)$$

$X > 0$; $\theta = U = 0$ as $Y \to \infty$;

$\tau^* \leq 0$:

$U = V = \theta = 0$ for $X > 0$ and $Y \geq 0$.

The condition on θ for $\tau^* > 0$ is numerically specified over a small finite region $Y \leq d$, where d is a dimensionless radius small in comparison to the values of X studied. The limiting condition employed analytically for a point heat source is $d \to 0$, with $t_0 \to \infty$, and similarity solutions have been obtained [9, 10]. Physically, the problem refers to a small disc, at temperature t_0, that gives a finite input of convected thermal energy Q which is conserved downstream. The input condition, as given above, is in terms of a specified temperature at $X = 0$.

The governing equations for an axisymmetric jet are the same as those for an axisymmetric plume in terms of the physical variables, equation (1). However, the characteristic velocity is now the velocity at the jet outlet U_0, which is taken as uniform across the small finite dimensionless radius d of the jet. The temperature at the outlet is taken as t_0. The dimensionless variables employed for this mixed convection problem are, therefore, different from those employed for the pure natural convection problem of a plume. These are:

$$X = x/\bar{X},\ Y = \frac{Y}{\bar{X}}\sqrt{Re},\ \theta = (t - t_{\infty,x})/(t_0 - t_{\infty,0}) \quad (5a)$$

$$U = u/U_0,\ V = \frac{v}{U_0}\sqrt{Re},\ \tau^* = \tau(U_0/\bar{X}),$$

with

$$Re = U_0\bar{X}/\nu,\quad Gr = g\beta\Delta t\,\bar{X}^3/\nu^2, \quad (5b)$$

$$\Delta t = t_0 - t_{\infty,0},\quad \tilde{Y} = \frac{y}{x}\sqrt{\frac{U_0 x}{\nu}} = Y/\sqrt{X}$$

where $\bar{X}$ is the vertical location up to where the velocity and temperature distributions are to be obtained. The governing equations then become:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{V}{Y} = 0, \quad (6a)$$

$$\frac{\partial U}{\partial\tau^*} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \left(\frac{Gr}{Re^2}\right)\theta + \frac{\partial^2 U}{\partial Y^2} + \frac{1}{Y}\frac{\partial U}{\partial Y}, \quad (6b)$$

$$\frac{\partial\theta}{\partial\tau^*} + U\frac{\partial\theta}{\partial X} + V\frac{\partial\theta}{\partial Y} + \frac{U}{\Delta t}\frac{\mathrm{d}t_{\infty,x}}{\mathrm{d}x} = \frac{1}{Pr}\left(\frac{\partial^2\theta}{\partial Y^2} + \frac{1}{Y}\frac{\partial\theta}{\partial Y}\right) \quad (6c)$$

with the boundary and initial conditions:

$\tau^* > 0$:

$X = 0$; $U = \theta = 1.0$ for $Y \leq d$, and $U = \theta = 0$ for $Y > d$

$$X > 0;\ \frac{\partial U}{\partial Y} = \frac{\partial\theta}{\partial Y} = V = 0 \quad \text{at } Y = 0,$$

$$X > 0;\ U = \theta = 0 \quad \text{as } Y \to \infty, \quad (7)$$

$\tau^* \leq 0$:

$U = V = \theta = 0$ for $X > 0$ and $Y \geq 0$.

If interest lies in the region close to $x = 0$, the characteristic length may be taken as d [12]. Here, we are largely interested in flow far downstream and, therefore, the vertical location $\bar{X}$ is employed for nondimensionalization. As $Y \to 0$, certain terms become indeterminate in the governing equations. To avoid this, L'Hôpital's rule is applied, along with the boundary conditions that apply at the centerline. Special equations are obtained, which are employed for determining U and θ at the centerline. The momentum equation for the plume, for instance, reduces to:

$$\frac{\partial U}{\partial\tau^*} + U\frac{\partial U}{\partial X} = \theta + 2\frac{\partial^2 U}{\partial Y^2}. \quad (8)$$

The governing momentum equation, given above for the two axisymmetric flows considered, is coupled to the energy equation through the buoyancy term. These equations, along with the continuity equation, are solved with the corresponding boundary and initial conditions by finite difference techniques. Though the primary goal is to obtain the steady state solution, the problem is solved as an unsteady state problem so as to observe the convergence of the solution to the required steady state as time elapses and to obtain better stability and convergence characteristics for these homogeneous boundary conditions [15]. The solution is obtained by employing a grid in the region of interest and marching in time with an explicit numerical scheme. Upwind differencing was employed for the convection terms. The grid spacing was varied to ensure negligible dependence of the results obtained upon the spacing employed. The specification of the initial conditions is arbitrary for the steady state solution and this was confirmed by varying them. The computing time for attaining steady state was found to vary with the initial conditions but the converged steady state solution was essentially independent of the specified conditions.

The numerical solution is obtained for a range of governing parameters. The dimensionless parameter d was taken as 1.0 for the plume and 10^{-3} for the jet, both being 0.1% of the range of X considered. The value of d was varied and it was found that the results, in physical terms, were independent of d, for a given thermal and momentum input, for $X \gtrsim 10d$. The x-dependence of the centerline velocity and temperature was also found to be independent of d far from the input location. Therefore, a finite value of d may be chosen so that it is much smaller than the values of X considered. The numerical results presented here are for the d values mentioned above and a comparison with earlier work on these flows for unstratified media indicated a close agreement, as discussed later.

For the explicit procedure employed, the time step $\Delta\tau^*$ is restricted due to stability considerations. For the momentum equation governing the plume flow, this gives $\Delta\tau^* \leq 1/[U/\Delta X + |V|/\Delta Y + 2/(\Delta Y)^2]$. Similar constraints arise for the other equations and the time step is chosen accordingly. The convergence criterion employed for obtaining the steady state solution is of the form $|\theta_{ij}^{n+1} - \theta_{ij}^{n}|_{max} \leq \varepsilon$, where the superscripts refer to the number of time steps and the subscripts to the location. The value of ε was varied to ensure that the solution obtained was independent of its value, which was finally chosen as 10^{-4}. The problem was solved on a DEC-1090 computer and some of the characteristic results are presented below.

NUMERICAL RESULTS AND DISCUSSION

Numerical results were obtained for the Prandtl number values of 0.7 and 7.0 and for various values of the other governing parameters. Various ambient temperature distributions were considered [14], though attention was focused largely on the linear variation. The basic trends and characteristics observed were similar for different distributions. The results presented here are for the linear variation unless indicated otherwise. The ambient stratification is characterized by the stratification parameter S given as:

$$S = \frac{d(t_{\infty,x}/\Delta t)}{dX}. \tag{9}$$

Therefore, for a linear ambient temperature variation, S is a constant and for other variations it is a function of X. For an ambient temperature variation of the form $(t_{\infty,x} - t_{\infty,0}) \propto X^n$, S may be written as S_0X^{n-1} The flow in an axisymmetric plume depends upon on only two parameters, S and Pr, and the results are obtained for various values of S. Figure 1 shows the temperature profiles for S ranging from 0, the unstratified circumstance, to 0.4. The results are shown for $Pr = 7.0$ at $X = 100$ and the local temperature excess is nondimensionalized with the centerline temperature excess over the ambient temperature, $t_c - t_{\infty,x}$, at the given value of X. The dimensionless temperature varies from 1.0 at the centerline to 0.0 away from the centerline, with zero slope at $Y = 0$, as specified. It is interesting to note that, though the profile does not vary significantly with S at low values of Y, the dimensionless temperature at large Y attains increasingly negative values with increasing S. Negative values as high as 8% of the centerline temperature excess are obtained at $S = 0.4$ and the boundary layer thickness increases at these negative values. The effect of stratification is, therefore, largest away from the centerline.

The appearance of negative values in the outer region of the boundary layer indicates a temperature defect in the flow. The ambient temperature increases with height and at a given vertical location, the fluid coming from below and in the outer region of the boundary layer is colder than the local ambient temperature. If the ambient temperature rise is sufficiently rapid, the fluid from below may not attain values higher than the local ambient temperature and a defect would result. This defect is a function of S, the Prandtl number and the vertical location X, increasing as the flow proceeds downstream. At low values of S and of X, the temperature defect was not observed, but the boundary layer thins with increasing S due to the decrease in the dimensionless temperature in the outer portion of the boundary layer. With a further increase in S, temperature defect arises and the boundary layer thickens due to the negative region obtained. This effect is similar to that observed in the similarity solution for vertical surfaces [3, 5].

A comparison of the results obtained with those of Mollendorf and Gebhart [10] for the unstratified circumstance is also made in Fig. 1 and a close agreement is observed. The corresponding curve for $S = S_0X^{n-1}$, where $S_0 = 0.4$ and $n = 0.2$, is also shown in Fig. 1. As expected, the effect on the temperature distribution is smaller in this case, as compared to that for the linear distribution, since the ambient temperature increases more gradually in this case.

Figure 2 shows the velocity profiles for an axisymmetric thermal plume in a linearly stratified environment at various values of S and at $Pr = 7.0$ and $X = 100$. With increasing S, the velocity level is seen to decrease, indicating the curbing effect on the flow due

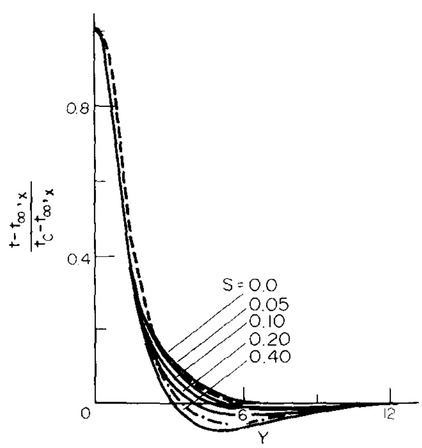

FIG. 1. Temperature profiles for an axisymmetric plume at $Pr = 7.0$ and $X = 100$, for various values of the stratification parameter S. --- temperature profile at $S = 0$ from [10]; -·- temperature profile for ambient temperature variation as $X^{0.2}$, with $S_0 = 0.4$.

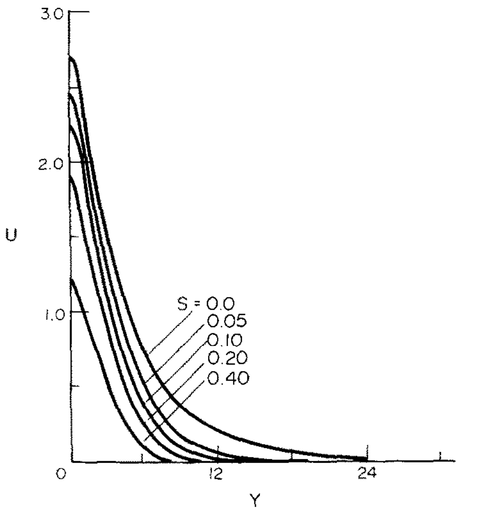

FIG. 2. Velocity profiles for an axisymmetric plume at Pr = 7.0 and X = 100, for various values of the stratification parameter S.

to a stable stratification. A comparison of the curve at $S = 0$ in Fig. 2 with the results obtained earlier for point source axisymmetric plumes was also made by converting the dimensionless variables to those employed by Mollendorf and Gebhart [10]. A close agreement was again obtained. The effect of increasing S is seen in a thinning of the boundary layer due to reduction of the velocity in the outer region of the flow. The reduction in the velocity is due to the decrease in buoyancy which accompanies the decrease in temperatures observed in Fig. 1. However, negative values are not observed over the range of S considered, indicating that the temperature defect is not strong enough to overcome the viscous and inertia forces to cause flow reversal. Again, the effect was found to increase downstream, giving flow reversal at larger X. Similar trends are seen in Figs. 3 and 4 for Pr = 0.7.

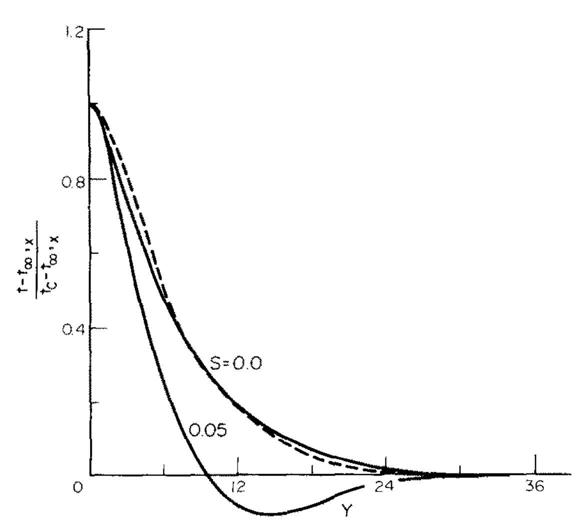

FIG. 3. Computed temperature profiles for the plume flow at Pr = 0.7 and X = 100. --- temperature profile at S = 0 from [10].

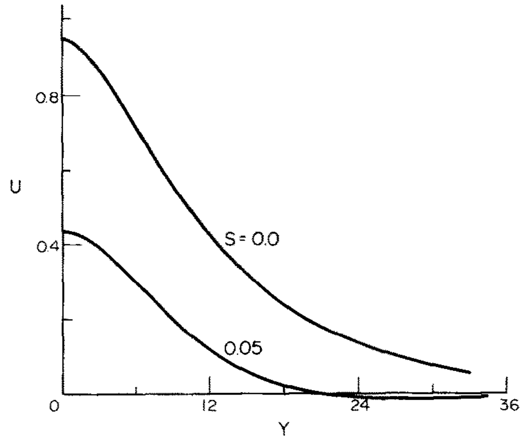

FIG. 4. Computed velocity profiles for the plume flow at Pr = 0.7 and X = 100.

The effect of stratification on the temperature field and on the flow is seen to be larger at the lower Prandtl number in terms of the dimensionless variables employed. A slight flow reversal is also observed in Fig. 4. A comparison of the temperature profile at $S = 0$ with that obtained by Mollendorf and Gebhart [10] again indicates a close agreement.

The downstream variation of the centerline temperature θ_c, where $\theta_c = (t_c - t_{\infty,x})/(t_0 - t_{\infty,0})$, is shown in Fig. 5. This figure, therefore, indicates the decay of the buoyancy level downstream and may be employed with Figs. 1 and 3 to determine the temperature variations in the flow with respect to the initial temperature difference at $X = 0$. For the unstratified case, the temperature excess was found to decay as $1/X$, which is predicted by the analysis of a point source plume. The decay is obviously faster for a stratified medium due to the ambient temperature rise. Zero temperature excess is also obtained at a downstream location, which is found to be at a lower X for a larger value of S, as physically expected. Beyond this downstream location, negative buoyancy arises and tends to stop the flow and cause flow reversal, thus

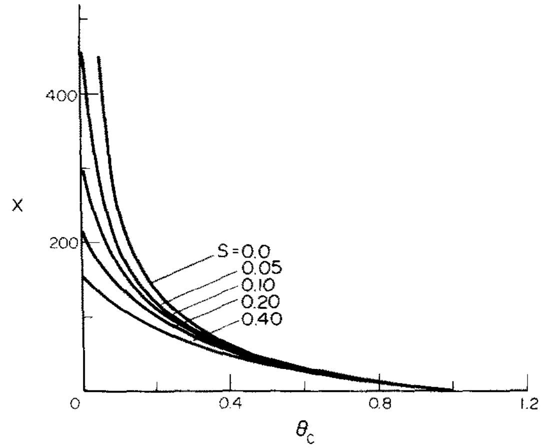

FIG. 5. Downstream centerline temperature variation in the plume at various values of S for Pr = 7.0.

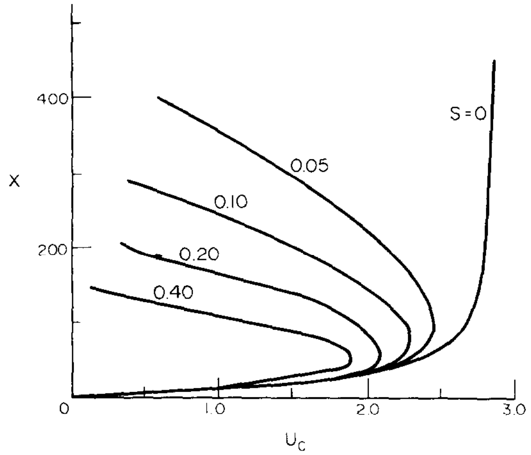

FIG. 6. Downstream variation of the centerline velocity in the plume at various values of S for Pr = 7.0.

restricting the height to which the flow rises. With increasing S, the height of plume rise, therefore, decreases.

This effect is seen more clearly in Fig. 6, where the downstream variation of the centerline velocity is shown. The curve for an unstratified medium, $S = 0$, indicates a sharp increase in the centerline velocity from zero to a constant value. Earlier analyses of a point source plume have also indicated a constant centerline velocity downstream, due to the decaying temperature level and increasing flow rate [9, 10]. However, in a stratified medium, the buoyancy level decreases at a faster rate downstream. Therefore, the constant velocity level of an unstratified environment is not maintained and U_c decreases downstream, approaching zero as the buoyancy becomes negative. The flow is, therefore, predicted to come to rest and reverse, dropping back to the zero buoyancy level, as physically expected [1, 7]. The flow in the region close to where zero buoyancy arises and flow reverses is not a boundary layer problem and the present study is not extended to a study of these effects. However, the anticipated trends are seen here.

In all the curves for non-zero S, a maximum velocity is observed, followed by a sharp decrease downstream. The X location where the maximum occurs varies only weakly with S. The zero value at $X = 0$ is due to the boundary conditions which represent a thermal energy input, with no momentum input. At low X, the buoyancy level is high and causes the flow to accelerate. The entrainment and decreasing buoyancy level that arise downstream then result in the observed decrease in the centerline velocity. This is a very interesting behavior and was found to be quite similar at the smaller Prandtl number too.

The flow in a buoyant axisymmetric jet in a stratified environment is governed by equation (6), which contains Gr/Re^2, Pr and S as parameters. The temperature profiles obtained at $Gr/Re^2 = 0.47$ and $X = 0.2$ for $Pr = 0.7$ are shown in Fig. 7 for a range of the stratification parameter S. Again, a temperature defect is observed in the outer region of the flow. With increasing S, the dimensionless temperature level decreases in the flow, thereby lowering the buoyancy. This affects the flow, as seen in Fig. 8. The velocity level decreases in the boundary layer and, though no flow reversal arises at this value of X, the boundary layer thins with increasing S. Similar trends are observed at $Pr = 7.0$ and at other values of the governing parameters. At a given location, the velocity level increases with an increase in the mixed convection parameter Gr/Re^2. At the larger Pr, the effect of stratification was smaller, as found earlier. The effect of stratification was again found to increase as the flow proceeds downstream. The temperature profiles at $Gr/Re^2 = 0.47$, $X = 0.1$ and $Pr = 7.0$ are seen in Fig. 9, indicating the smaller effect of stratification at the larger Pr.

The downstream variation of the centerline tem-

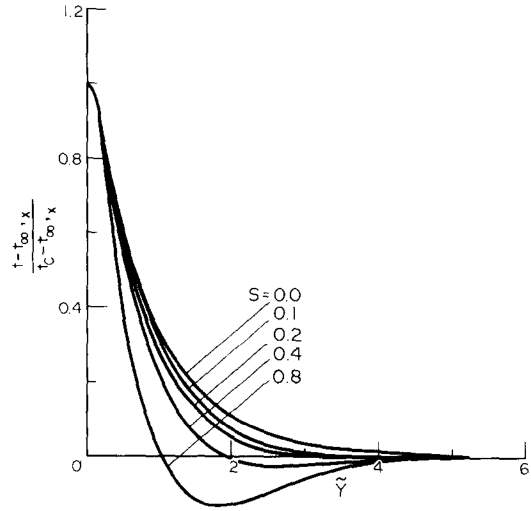

FIG. 7. Temperature distributions in an axisymmetric jet at $Pr = 0.7$, $X = 0.2$ and $Gr/Re^2 = 0.47$ for various stratification levels.

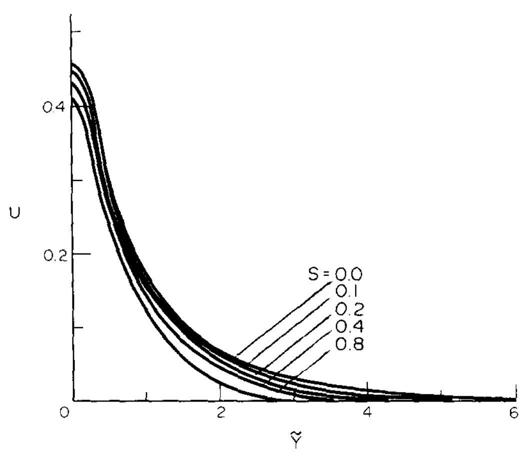

FIG. 8. Velocity distributions in the axisymmetric jet flow at $Pr = 0.7$, $X = 0.2$ and $Gr/Re^2 = 0.47$ for various values of S.

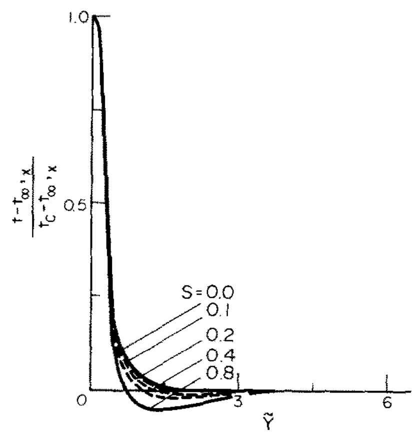

FIG. 9. Temperature profiles in the jet flow at Pr = 7.0, X = 0.1 and Gr/Re^2 = 0.47.

perature and velocity are shown in Figs. 10 and 11 at Gr/Re^2 = 0.47 and Pr = 0.7. Both the centerline temperature and velocity decrease downstream. The unstratified case corresponds to the buoyant jet flow considered in [11–13] and the observed trends are close to those obtained earlier. With increasing S, the temperature difference decreases more rapidly. This is also reflected in the downstream variation of the velocity, as shown in Fig. 11. The dimensionless centerline velocity starts at 1.0 at X = 0 and as the flow proceeds downstream, entrainment of ambient fluid results in a decreasing velocity. When the medium is stably stratified, with the ambient temperature increasing with height, buoyancy decreases downstream, resulting in a velocity decrease which is more rapid than that for an unstratified medium. Again, when buoyancy becomes zero and then negative downstream, the flow is expected to reverse as discussed earlier for a plume. For the unstratified case, this circumstance obviously does not arise, as is also seen from the trends indicated in Fig. 11. The initial decay in velocity and in temperature is very rapid, with a more

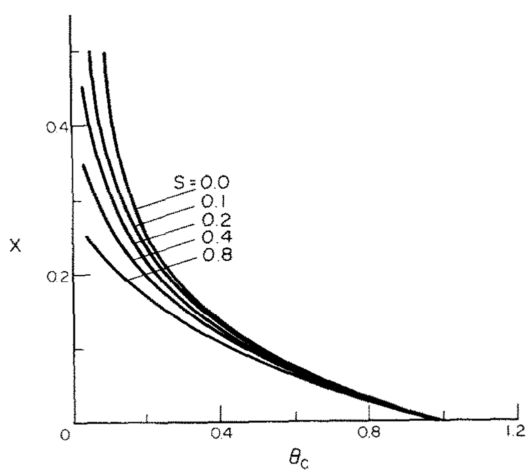

FIG. 10. Downstream centerline temperature variation in the buoyant jet flow at Pr = 0.7 and Gr/Re^2 = 0.47.

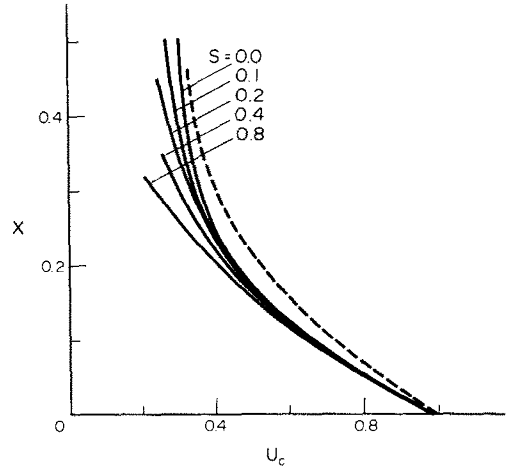

FIG. 11. Downstream variation of the centerline velocity in the buoyant jet at Pr = 0.7 and Gr/Re^2 = 0.47. --- variation at S = 0 for Gr/Re^2 = 0.94.

gradual variation further downstream. The basic features of the flow are similar to those observed for turbulent free boundary flows in a stably stratified environment [1, 7]. The curve for S = 0 and Gr/Re^2 = 0.94 is also shown in Fig. 11, indicating a higher velocity level due to the increased buoyancy effects in the flow. Very far downstream, the flow tends to behave as a plume due to decreasing starting effects. This trend is seen in the approach of the centerline velocity towards a constant value, which is the predicted result for an axisymmetric plume in an isothermal medium. Unfortunately, no experimental results exist for these laminar flows in stratified media, though work has been done on turbulent flows. However, the same analysis was applied to 2-dim. plumes and a fairly good agreement with experimental results was obtained [14].

CONCLUDING REMARKS

An analytical study of laminar free boundary buoyancy-induced flows in a stably stratified medium has been carried out. The axisymmetric thermal plume and the buoyant jet are considered in a temperature stratified environment, with the input temperature and velocity specified over a small finite region. The velocity and temperature profiles are studied downstream in the two cases, for various thermal stratification levels, for two Prandtl numbers, 0.7 and 7.0, and for various values of other governing parameters. Of particular interest were the downstream variation of the centerline velocity and temperature and the effect of increasing stratification on the profiles downstream.

The existence of thermal stratification is found to lower the local buoyancy, giving rise to significant negative buoyancy downstream at large stratification levels, particularly in the outer region of the flow. This results in a substantial lowering of the velocity level and flow reversal at large values of the stratification parameter and/or at large downstream distances. The results obtained indicate the approach to zero velocity at the centerline as the buoyancy level decreases below zero. A maximum velocity is observed downstream for a plume, followed by a rapid decrease, in stratified media. A uniform decrease is observed in a buoyant jet. The effect of stratification on the flow is found to be larger at the smaller Prandtl number. The results for the unstratified case are found to agree with earlier similarity results for the point source free boundary flows. For a stratified medium, the results are similar in some respects to the earlier studies of other natural convection flows. But several interesting differences and characteristics, relevant to the flows considered in this study, are brought out. The observed behavior is also considered in terms of the underlying physical processes.

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ECOULEMENTS LAMINAIRES NATURELS, AXISYMETRIQUES A FRONTIERE LIBRE DANS UN MILIEU THERMIQUEMENT STRATIFIE

Résumé—On étudie analytiquement l'écoulement laminaire et axisymétrique dans les panaches thermiques et les jets libres dans les milieux stratifiés. On considère en détail une variation linéaire de la température ambiante, avec une entrée d'énergie thermique et de quantité de mouvement spécifiées sur une petite région finie. L'écoulement en aval est étudié et des résultats numériques sont obtenus pour un large domaine de niveaux de stratification et pour plusieurs valeurs des paramètres. Un intérêt particulier est porté sur la variation de la vitesse et de la température le long de l'axe. Les profils correspondants sont obtenus. On trouve qu'une stratification stable cause une décroissance dans l'effet de vitesse. Un défaut de température apparait dans la région externe de la couche limite et la vitesse dans l'écoulement diminue sensiblement. L'effet augmente avec la distance en aval et avec un accroissement du niveau de stratification. L'effet d'Archimède décroit en aval et les résultats observés indiquent la nature de l'écoulement dans la région où les forces d'Archimède s'annulent puis deviennent négatives. On considère le mécanisme physique à partir des résultats obtenus.

DURCH AUFTRIEB INDUZIERTE LAMINARE ACHSENSYMMETRISCHE STRÖMUNGEN OHNE BERGRENZUNG IN EINEM THERMISCH GESCHICHTETEN MEDIUM

Zusammenfassung—Es wird eine analytische Untersuchung der laminaren Strömung in achsensymmetrischen thermischen Auftriebsströmungen und Auftriebsstrahlen in stabil geschichteten Medien durchgeführt. Eine lineare Variation der Umgebungstemperatur wird im einzelnen betrachtet, wobei ein Zustrom von thermischer Energie und Impuls, der innerhalb eines kleinen begrenzten Gebietes definiert ist, stattfindet. Die Strömung stromabwärts wird im einzelnen untersucht. Es werden numerische Ergebnisse für einen weiten Bereich des Schichtungsniveaus und der bestimmenden Parameter erzielt. Von besonderem Interesse waren die stromabwärts auftretende Veränderung der Geschwindigkeit und Temperatur in der Mittellinie. Die zugehörigen Profile wurden ebenfalls erhalten. Es wurde festgestellt, daß eine stabile Schichtung eine Abnahme des Auftriebs und dadurch der Geschwindigkeit verursacht. Im äußeren Bereich der Grenzschicht ergibt sich ein Temperatursprung, und das Geschwindigkeitsniveau der Strömung nimmt ebenfalls stark ab. Es wurde festgestellt, daß dieser Effekt mit dem Abstand stromabwärts und mit ansteigendem Schichtungsniveau zunimmt. Der Auftrieb nimmt in Strömungsrichtung ab, und die beobachteten Ergebnisse zeigen die Strömungsstruktur in dem Bereich, in dem die Auftriebskräfte null und dann negativ werden. Die zugrundeliegenden physikalischen Vorgänge werden im Rahmen der erzielten Ergebnisse betrachtet.

ЛАМИНАРНЫЕ СВОБОДНОКОНВЕКТИВНЫЕ ОСЕСИММЕТРИЧНЫЕ ПОГРАНИЧНЫЕ ТЕЧЕНИЯ В ТЕРМИЧЕСКИ СТРАТИФИЦИРОВАННОЙ СРЕДЕ

Аннотация — Выполнено аналитическое исследование ламинарного течения в осесимметричных тепловых факелах и свободноконвективных струях в устойчиво стратифицированных средах. Подробно рассмотрено влияние линейного изменения температуры окружающей среды при заданных значениях плотности подводимых потоков тепла и импульса, сосредоточенных в небольшой конечной области. Проведено подробное исследование опускного течения, и получены численные результаты для широкого диапазона степени стратификации и различных значений основных параметров. Особое внимание уделено осевому изменению скорости и температуры в опускном потоке. Также получены соответствующие профили. Найдено, что устойчивая стратификация снижает величину подъемной силы, и, следовательно, скорости. Температура уменьшается во внешней области пограничного слоя, что приводит к существенному снижению скорости потока. Показано, что эффект усиливается при продвижении вниз по потоку и увеличении степени стратификации. Величина подъемной силы уменьшается с увеличением расстояния вниз по потоку. Полученные результаты используются для объяснения течения в области, где подъемная сила принимает нулевое, а затем отрицательное значение. Рассмотрены лежащие в основе этих явлений физические процессы.